

Lecture 4

Continuous-Time Fourier Transform in One Dimension

4.1 Learning Objectives

- Understand the concept of the Fourier Transform and the idea of representing a signal (image) in the spatial domain versus in the spatial frequency domain.
- Understand the concept of moving back and forth between the two domains (spatial domain versus in the spatial frequency domain), ie: Fourier transform and inverse Fourier transform.
- Recognize a few of the widely used Fourier transform pairs.

4.2 Introduction to Signal Analysis

Signal analysis and synthesis are foundational components of signal processing, with a wide variety of applications. These techniques can be described as follows¹

Signal analysis: The original signal is analyzed by breaking down the signal into elementary components that describe the “characteristic properties” of this particular signal

Signal synthesis: If we know the elementary components of the original signal, how do we use these elementary components to reconstruct the signal, or at least construct a reasonable approximation which is “close” to the original signal in some sense.

One central question in the field of signal analysis and synthesis is the choice of these elementary components that we use to break down our signals of interest. In Fourier

¹<http://web.abo.fi/fak/mmf/mate/kurser/krusningar/chap1.pdf>

analysis, the classical analysis method in signal processing, these elementary components are sinusoidal functions (sines and cosines).

Signal analysis techniques have enormous applications in a wide array of signal processing scenarios, including a variety of applications in medical imaging. Specific applications of signal analysis include denoising, filtering, compression, solution of certain differential equations, image reconstruction, image processing, as well as to visualize signals in various forms of spectroscopy (eg: NMR).

4.3 Preliminaries: the Fourier Series

The Fourier Transform can be traced back to the Fourier Series, which provides a way to represent periodic functions as a sum (infinite sum) of sines and cosines. For a periodic function $f(x)$ with period L , its Fourier Series decomposition is as follows:

$$f_N(x) = a_0 + \sum_{n=1}^N (a_n \cos(2\pi nx/L) + b_n \sin(2\pi nx/L)) \quad (4.1)$$

where N is the (potentially infinite) number of terms in the sum. A common example to illustrate the effect of the Fourier series is to consider a square wave with period 1:

$$f(x) = \begin{cases} -1 & , \text{ for } -0.5 \leq x < 0 \\ 1 & , \text{ for } 0 \leq x < 0.5 \end{cases} \quad (4.2)$$

which is then repeated for values of x outside the interval $[-0.5, 0.5)$. This signal could be roughly approximated by a single sine wave with period 1, ie: $f(x) \approx \frac{4}{\pi} \sin(2\pi x)$ (where the constant scaling $\frac{4}{\pi}$ helps fit the amplitude a bit better). Obviously, this sinusoidal approximation is not very accurate, as it only has low-frequency components and is not able to capture the sharp edges in the original square wave. However, we can improve our approximation by including additional sinusoids (higher N) with higher spatial frequency, but lower amplitude, as follows:

$$f(x) \approx \frac{4}{\pi} \sin(2\pi x) + \frac{1}{3} \frac{4}{\pi} \sin(2\pi \cdot 3 \cdot x) + \frac{1}{5} \frac{4}{\pi} \sin(2\pi \cdot 5 \cdot x) + \dots \quad (4.3)$$

This example is shown graphically in Figure [4.1](#).

In general, the coefficients a_n and b_n in the Fourier series expression (Eq. [4.1](#)) can be calculated as:

$$a_0 = \frac{1}{L} \int_{-L/2}^{L/2} f(x) dx \quad (4.4)$$

$$a_n = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \cos(2\pi nx/L) dx, \text{ for } n = 1, 2, \dots \quad (4.5)$$

$$b_n = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \sin(2\pi nx/L) dx, \text{ for } n = 1, 2, \dots \quad (4.6)$$

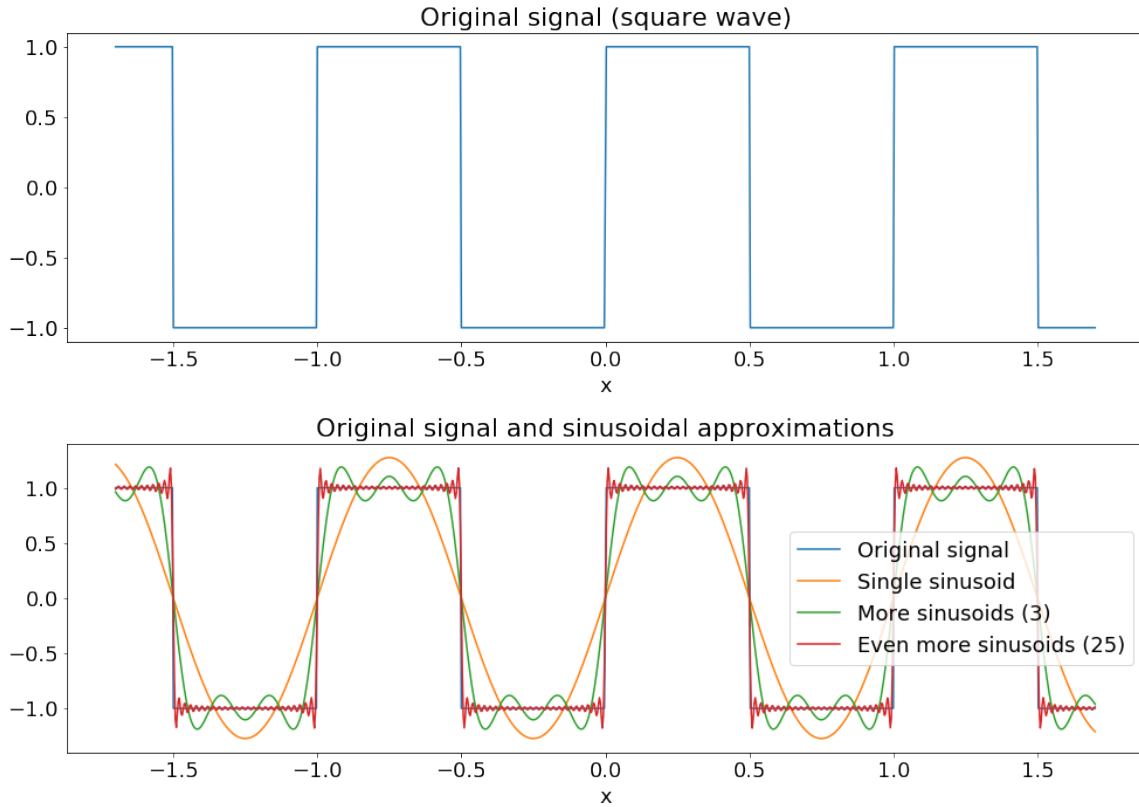


Figure 4.1: Illustration of the Fourier series to represent a periodic function. A square wave can be represented with increasing accuracy by including more and more sinusoids of increasing frequency.

Question: triangle wave. Let us consider a triangle wave $f(x)$ that is periodic with period L and amplitude A , and symmetric about the origin $x = 0$, ie: a wave with maximum value $f(x) = A$ reached at $x = 0 \pm mL$ for $n = \dots, -2, -1, 0, 1, 2, \dots$, and minimum value $f(x) = -A$ reached at $x = L/2 \pm mL$ for $n = \dots, -2, -1, 0, 1, 2, \dots$. What are the corresponding Fourier series coefficients for this triangle wave?

In this course, the sinusoidal decomposition of the Fourier series (and the Fourier transform; see below) is more conveniently expressed in terms of complex exponentials. In this version, the Fourier series decomposition of a periodic function $f(x)$ with period L can be written as:

$$f_N(x) = \sum_{n=-N}^N c_n e^{i2\pi nx/L} \quad (4.7)$$

where N (which is potentially infinite) determines the number of terms included in the sum, and the coefficients c_n can be calculated as:

$$c_n = \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-i2\pi nx/L} dx \quad (4.8)$$

Convergence: We have provided a definition of the Fourier series $f_N(x)$, as well as a formula to evaluate its coefficients. However, we still have not defined precisely what happens with $f_N(x)$ as $N \rightarrow \infty$ (e.g., whether it converges, and what it converges to). Convergence is rather benign for the kinds of signals we cover in this course. For example, for periodic signals $f(x)$ that are square integrable over the period of length L , the Fourier series $f_N(x)$ converges to $f(x)$ in the L_2 sense, ie:

$$\lim_{N \rightarrow \infty} \|f_N(x) - f(x)\|_2 = \lim_{N \rightarrow \infty} \sqrt{\int_{-L/2}^{L/2} |f_N(x) - f(x)|^2 dx} = 0$$

Note that this L_2 convergence does not imply pointwise convergence (i.e., convergence for every value of x). The most important illustration of this distinction is the presence of discontinuities (jumps) in the function $f(x)$. This leads to an effect termed “Gibbs ringing”, which can actually be appreciated in Figure 4.1 above (notice the way the “even more sinusoids” signal approximates the original signal closely at all points except at the discontinuities). Gibbs ringing will be examined in more detail in this course, including below in this lecture. For most signals $f(x)$ relevant to medical physics and engineering, the Fourier series converges to the value of the signal as $N \rightarrow \infty$, at every point x where $f(x)$ is continuous. Mathematically, this can be guaranteed with mild conditions such as the Dirichlet conditions ($f(x)$ is absolute integrable over a period, has bounded variation within any bounded interval, has a finite number of discontinuities in any bounded interval, and these discontinuities are themselves finite).

Non-periodic functions: What happens as the period becomes longer and longer and approaches $L \rightarrow \infty$?. As L grows, the sum in the Fourier series “becomes” an integral and leads to the Fourier transform. The Fourier transform is defined below, and will be the focus of this course for the next few lectures.

4.4 Definition of the Continuous Fourier Transform

Analogously to the Fourier series coefficients described above (Eq. 4.8), the Fourier Transform (FT) for a (not necessarily periodic) continuous signal $f(x)$ is defined as follows:

$$\hat{f}(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi x u} dx \quad (4.9)$$

Generally, this FT operation can be viewed as expressing a time-domain signal (if the variable x corresponds to time) in terms of its temporal frequency components, or a spatial-domain signal in terms of its spatial frequency components.

Similarly, the inverse Fourier Transform (iFT) can be expressed as follows:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(u) e^{i2\pi x u} du \quad (4.10)$$

which is essentially the same expression as the FT, except the sign on the exponential is flipped. Note that this expression is analogous to the Fourier series (Eq. 4.7), except replacing the sum by an integral. The signals $f(x)$ and $\hat{f}(u)$ are referred to as a Fourier Transform pair. Please see examples of common FT pairs below.

Note that the complex exponential $e^{i2\pi xu}$ can be written as a combination of a sine and cosine wave, ie:

$$e^{i2\pi xu} = \cos(2\pi xu) + i \sin(2\pi xu) \quad (4.11)$$

In other words, the FT can be seen as decomposing our signal $f(x)$ into a combination of sine and cosine waves.

As we will see throughout this course, the FT is a complex-valued operation: even if our initial signal $f(x)$ is real-valued, its FT $\hat{f}(u)$ is generally (although not always; see examples below) complex-valued.

Question: can you think about functions that appear easy to express with the Fourier Transform, ie: as a combination of sines and cosines?

Question: can you think about functions that appear difficult to express as a combination of sines and cosines?

4.5 A Few Common Fourier Transform Pairs

$f(x)$	$\hat{f}(u)$
$\delta(x)$	1
1	$\delta(u)$
$\delta(x - x_0)$	$e^{-i2\pi ux_0}$
$e^{i2\pi xu_0}$	$\delta(u - u_0)$
$e^{-\pi x^2}$	$e^{-\pi u^2}$
$\sin(2\pi u_0 x)$	$\frac{i}{2} [\delta(u + u_0) - \delta(u - u_0)]$
$\cos(2\pi u_0 x)$	$\frac{1}{2} [\delta(u + u_0) + \delta(u - u_0)]$
$\text{rect}(x)$	$\text{sinc}(u) = \frac{\sin(\pi u)}{\pi u}$
$\text{sinc}(x)$	$\text{rect}(u)$
$\text{III}(x)$	$\text{III}(u)$

Note: $\text{III}(x) = \sum_{m=-\infty}^{\infty} \delta(x - m)$ is the so-called “comb” function. As shown in the table above, the Fourier transform of a comb function is also a comb function. This function will be central to our coverage of sampling in a few lectures.

4.6 A Few Mathematical Questions

4.6.1 Does the Fourier Transform Integral Always Exist?

No. There are plenty of functions for which the integral in Equation [4.9](#) does not converge. A sufficient (but not necessary!) condition for the existence of the Fourier Transform is the absolute integrability of our signal:

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty \quad (4.12)$$

If this condition is met, then the Fourier Transform integral is guaranteed to converge. However, it is very important to note that some functions that are central to Fourier analysis (including sinusoids!) do not meet this criterion. Throughout this course, whenever we encounter these types of functions, we will typically be able to overcome the challenge by using δ functions, as shown in the Fourier Transform pairs table within the previous section.

4.6.2 What do we mean by ‘=’?

The definition of the inverse Fourier Transform given above (Equation [4.10](#)) indicates that we can express a function $f(x)$ as a combination (integral) of complex exponentials, where each complex exponential is weighted by a certain value (given by $\hat{f}(u)$). Further, the ‘=’ sign seems to indicate that if we start out with a certain signal and take the FT and then the iFT, we should end up with exactly the same signal.

However, let us consider a simple example:

$$f(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases} \quad (4.13)$$

Note that this is a bit of an unusual signal in the sense that its energy is zero (this is not a delta function!). Now, if we take the Fourier Transform, it will be zero everywhere, and if we then take the inverse Fourier Transform it will be zero too. However, our original signal $f(x)$ is not zero. This seems to contradict the ‘=’ sign in Equation [4.10](#).

This is just one example of an overall property of the Fourier Transform, which allows us to represent and recover many signals, up to differences which have zero energy (ie: the FT will not be able to distinguish a signal that is zero everywhere from our signal defined in Equation [4.13](#) above).

In the context of Fourier Transforms, this can be interpreted as permission to ignore zero-energy signals.

4.6.3 How Can the Fourier Transform Represent non-Sinusoidal Signals?

It is illustrative to take a closer look at how a decomposition into sines and cosines such as the Fourier Transform can represent signals that are clearly non-sinusoidal. An intuitive

way to think about it is to take a closer look at the expression for the inverse Fourier Transform, and limit it to a certain range of frequencies $[-B, B]$:

$$f_B(x) = \int_{-B}^B \hat{f}(u) e^{i2\pi x u} du \quad (4.14)$$

In this case, we can start out with a small value of B , and successively increase B in order to approximate the full expression of our signal $f(x)$ in terms of $\hat{f}(u)$. This process is illustrated in Figure [4.2](#), for the specific case of a rect function. Note how the approximation becomes better and better as we increase B . Indeed, our approximated $f_B(x)$ becomes closer and closer to f_B everywhere except near the sharp edges of the rect. This effect, known as Gibbs ringing, is a fundamental consequence of representing non-sinusoidal signals using sinusoids. Gibbs ringing extends easily to the multi-dimensional signal case, and has important implications for imaging. Some of these implications will be explored further throughout this course.

Question: can you think of an intuitive or graphical interpretation of the fact that the FT of a real-valued signal $f(x)$ generally results in a complex-valued signal $\hat{f}(u)$? What does the phase of $\hat{f}(u)$ mean? Hint: take a look at the “translation” property of the Fourier Transform in the following lecture (or look up this property online).

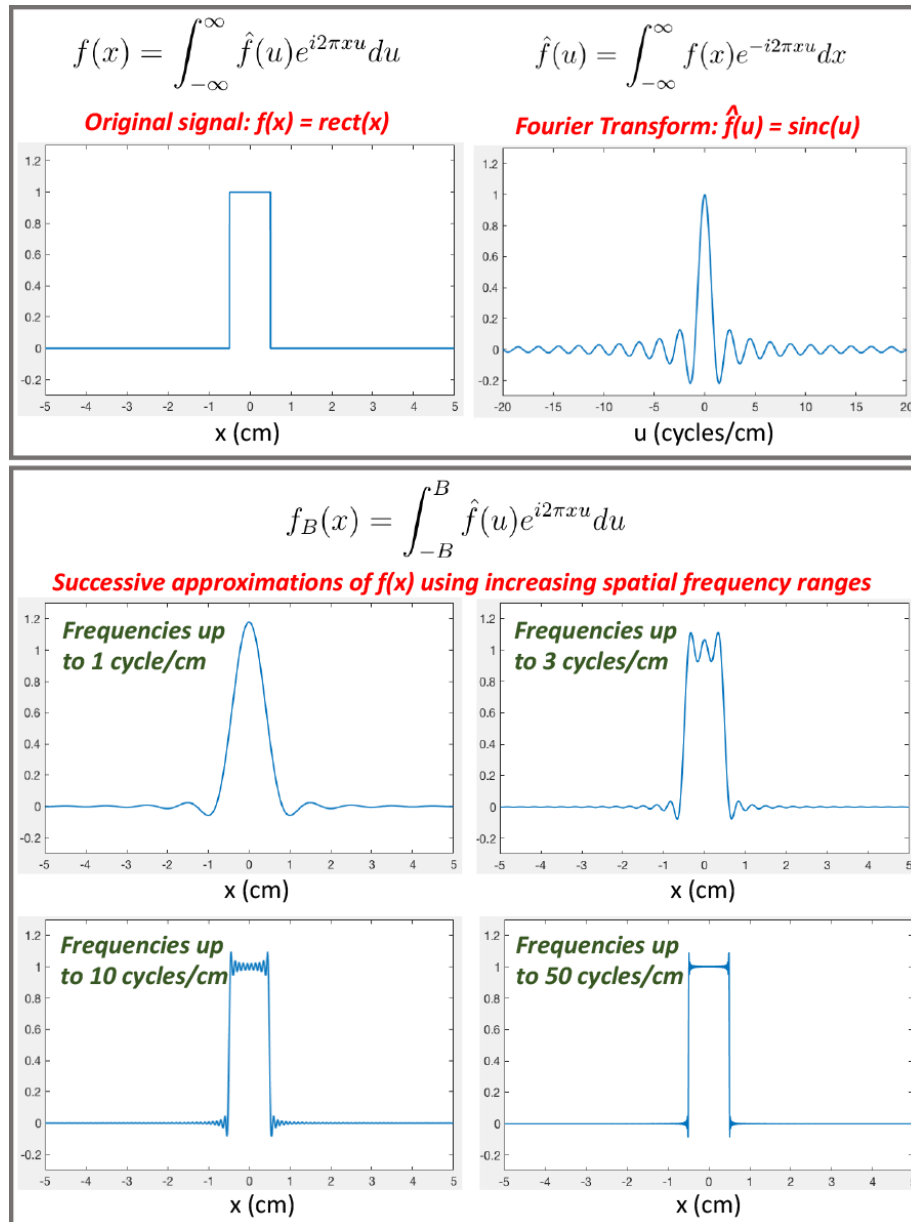


Figure 4.2: The Fourier Transform enables us to express non-sinusoidal functions (eg: a rect) as a combination of infinitely many sinusoids. However, if we restrict the range of sinusoid frequencies we use, we are not able to accurately represent certain features (eg: sharp edges). As we increase the frequency range, we obtain a better approximation to our original signal, everywhere except near the sharp edges. Near these sharp edges, we observe increasingly rapid and narrow oscillations, with a nearly constant amplitude (approximately 9% of the height of the edge), known as Gibbs ringing.