

Lecture 6

Fourier Transform in N Dimensions

6.1 Learning Objectives

- Develop an intuitive understanding for multidimensional spatial frequencies, and the N-D Fourier transform.
- Recognize the typical appearance of N-D Fourier transforms, eg: the distribution of signal in the spatial frequency domain.
- Recognize basic connections between features in the spatial domain versus the spatial frequency domain.

6.2 Introduction

The extension of the FT to multi-dimensional signals enables us to analyze images (or higher-dimensional signals) in terms of a combination of multi-dimensional complex exponentials (waves with various frequencies and ‘directions’).

6.3 Definition

In the general case of an N-dimensional space, we consider a signal $f(\mathbf{r})$, where $\mathbf{r} = [r_1, r_2, \dots, r_N]$ denotes a location in $\mathbb{R}^N$. The N-dimensional Fourier Transform (FT) of $f(\mathbf{r})$ is defined as:

$$\hat{f}(\mathbf{u}) = \int_{\mathbb{R}^N} f(\mathbf{r}) e^{-i2\pi \mathbf{r} \cdot \mathbf{u}} d\mathbf{r}$$

where $\mathbf{r} \cdot \mathbf{u} = r_1 u_1 + r_2 u_2 + \dots + r_N u_N$ is the dot product between vectors $\mathbf{r}$ and $\mathbf{u}$.

The corresponding inverse FT (iFT) is defined as:

$$f(\mathbf{r}) = \int_{\mathbb{R}^N} \hat{f}(\mathbf{u}) e^{i2\pi \mathbf{r} \cdot \mathbf{u}} d\mathbf{u}$$

Similar to the 1D FT case, the iFT expression above effectively describes the decomposition of our signal $f(\mathbf{r})$ into a superposition of planar waves $e^{-i2\pi\mathbf{r}\cdot\mathbf{u}}$, where each of these waves has spatial frequency given by $\mathbf{u}$. When combined to reconstruct $f(\mathbf{r})$, each of these planar waves is scaled by a complex-valued amplitude given by $\hat{f}(\mathbf{u})$.

Using a slightly different notation, this expression can be reduced to the specific case of a 2D image $f(x, y)$:

$$\hat{f}(u, v) = \int_{\mathbb{R}^2} f(x, y) e^{-i2\pi(xu+yv)} dx dy$$

with corresponding iFT given by:

$$f(x, y) = \int_{\mathbb{R}^2} \hat{f}(u, v) e^{i2\pi(xu+yv)} du dv$$

6.4 A Few Common Fourier Transform Pairs

$f(x, y)$	$\hat{f}(u, v)$
$\delta(x, y)$	1
1	$\delta(u, v)$
$\delta(x - x_0, y - y_0)$	$e^{-i2\pi(xu_0+yv_0)}$
$e^{i2\pi(xu_0+yv_0)}$	$\delta(u - u_0, v - v_0)$
$\text{rect}(x, y)$	$\text{sinc}(u)\text{sinc}(v) = \frac{\sin(\pi u)}{\pi u} \frac{\sin(\pi v)}{\pi v}$

6.5 Interpretation and Examples

As illustrated in Figure [6.1](#), the 2D FT represents a 2D signal (an image) as a superposition of 2D waves with different frequencies and orientations.

Additionally, the 2D FT enables insightful interpretations of the different 2D frequency regions. For instance the low frequency regions (close to $(u, v) = (0, 0)$) represent smooth features, whereas the high frequency regions represent sharp features and edges. This is illustrated in Figure [6.2](#).

Finally, image artifacts are sometimes easy to interpret in Fourier space. An example of a spike artifact is shown in Figure [6.3](#).

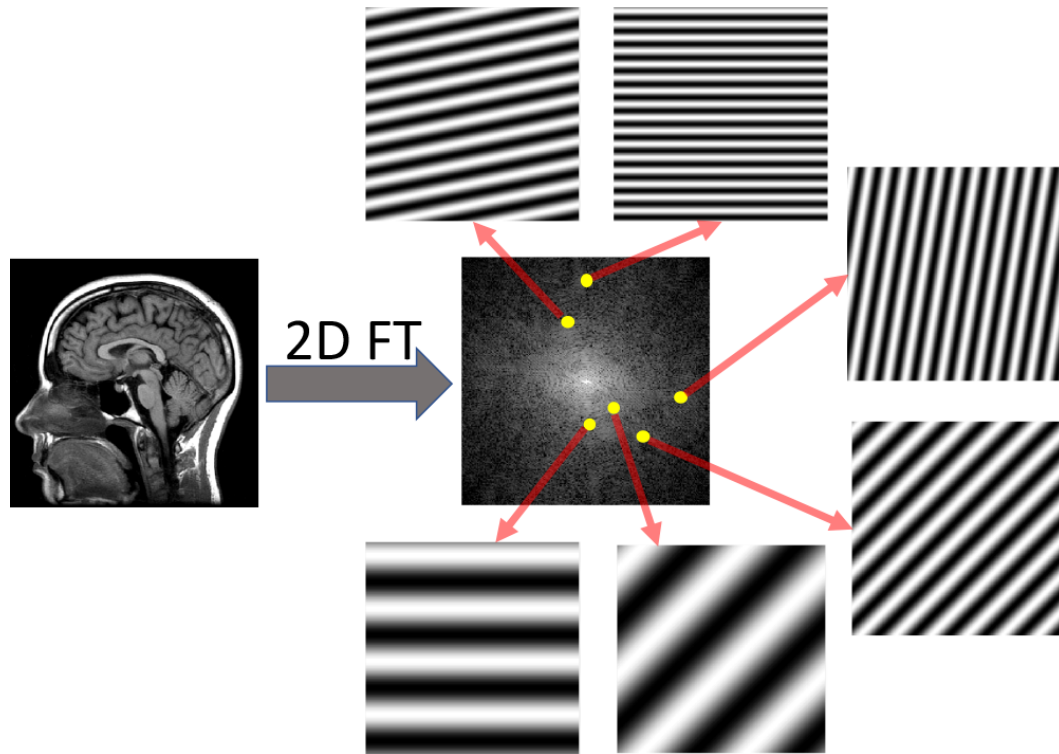


Figure 6.1: Fourier Transform in 2D applied to a brain MR image. This 2D FT results in the decomposition of the image in a set of 2D wave patterns, each weighted by a different amplitude. This amplitude is given by the 2D FT.

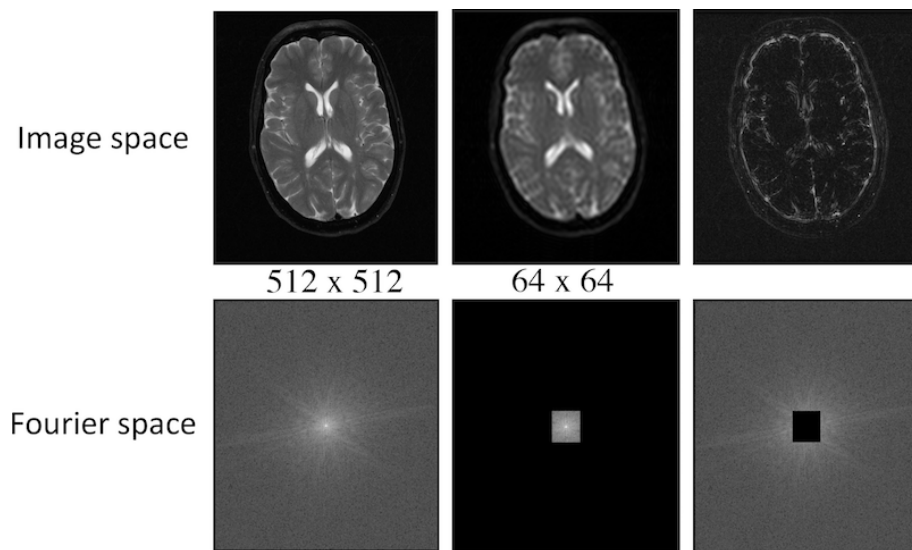


Figure 6.2: Separating the low and high frequency components of a 2D image. Note that the low frequency components correspond to smooth features, whereas the high frequency components correspond to sharp features and edges.

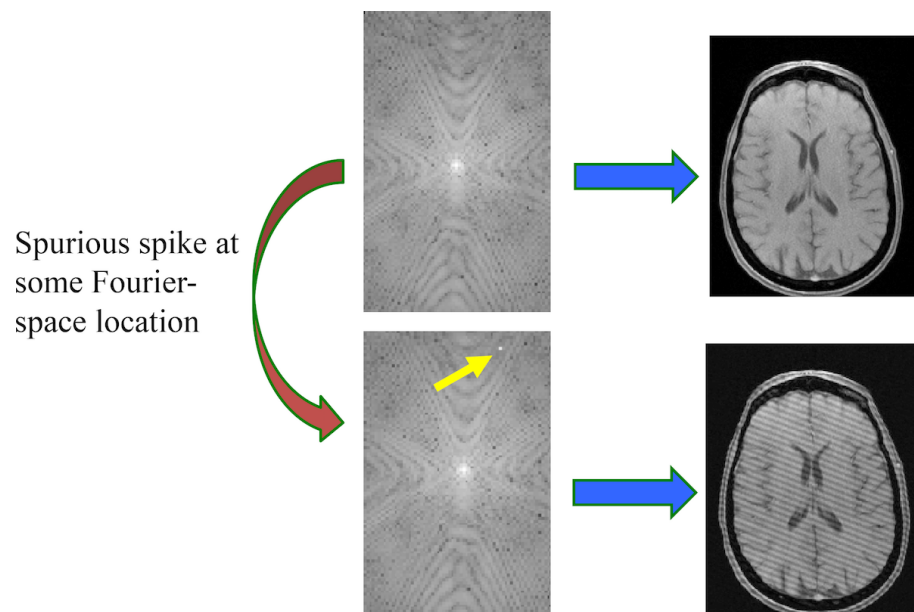


Figure 6.3: The presence of a spike (high localized signal) in the Fourier Transform of an image results in the addition of sinusoidal artifacts (sinusoids with spatial frequency given by the location of the spike in Fourier space) in the image space.