

Lecture 7

Properties of the N-D Fourier Transform

7.1 Learning Objectives

- Recognize the properties of the N-D Fourier transform that are simple extensions of the corresponding properties of the 1-D Fourier transform (eg: linearity).
- Recognize several properties of the N-D Fourier transform that are unique to the multi-dimensional case.

7.2 Introduction

The multi-dimensional FT inherits the properties of the 1D FT. Also, several important additional properties arise in the presence of higher dimensions. In this lecture, we will describe these properties.

7.3 Properties of the N-D Fourier Transform Analogous to the 1D Case

The N-D FT has a number of properties that are essentially the same as in the 1D case:

- *Linearity.* For any FT pairs $f_1(\mathbf{r}) \Leftrightarrow \hat{f}_1(\mathbf{u})$, and $f_2(\mathbf{r}) \Leftrightarrow \hat{f}_2(\mathbf{u})$, and complex scalars a_1 and a_2 , the FT of $f_3(\mathbf{r}) = a_1 f_1(\mathbf{r}) + a_2 f_2(\mathbf{r})$ is given by:

$$\hat{f}_3(\mathbf{u}) = a_1 \hat{f}_1(\mathbf{u}) + a_2 \hat{f}_2(\mathbf{u})$$

- *Translation.* For any signal $f_1(\mathbf{r})$ with FT $\hat{f}_1(\mathbf{u})$, the FT of its shifted version $f_2(\mathbf{r}) = f_1(\mathbf{r} - \mathbf{r}_0)$ is given by a modulated version of $\hat{f}_1(\mathbf{u})$:

$$\hat{f}_2(\mathbf{u}) = e^{-i2\pi\mathbf{r}_0 \cdot \mathbf{u}} \hat{f}_1(\mathbf{u})$$

- *Modulation.* For any signal $f_1(\mathbf{r})$ with FT $\hat{f}_1(\mathbf{u})$, the FT of its modulated version $f_2(\mathbf{r}) = e^{i2\pi\mathbf{r}\cdot\mathbf{u}_0} f_1(\mathbf{r})$ is given by a translated version of $\hat{f}_1(\mathbf{u})$:

$$\hat{f}_2(\mathbf{u}) = \hat{f}_1(\mathbf{u} - \mathbf{u}_0)$$

- *Conjugation.* For any signal $f_1(\mathbf{r})$ with FT $\hat{f}_1(\mathbf{u})$, the FT of its conjugated version $f_2(\mathbf{r}) = \overline{f_1(\mathbf{r})}$ is given by

$$\hat{f}_2(\mathbf{u}) = \overline{\hat{f}_1(-\mathbf{u})}$$

- *Parseval's Theorem.* For any FT pairs $f_1(\mathbf{r}) \Leftrightarrow \hat{f}_1(\mathbf{u})$, and $f_2(\mathbf{r}) \Leftrightarrow \hat{f}_2(\mathbf{u})$ the following equality holds:

$$\int_{\mathbb{R}^N} f_1(\mathbf{r}) \overline{f_2(\mathbf{r})} d\mathbf{r} = \int_{\mathbb{R}^N} \hat{f}_1(\mathbf{u}) \overline{\hat{f}_2(\mathbf{u})} d\mathbf{u}$$

- *Convolution Theorem.* For any FT pairs $f_1(\mathbf{r}) \Leftrightarrow \hat{f}_1(\mathbf{u})$, and $f_2(\mathbf{r}) \Leftrightarrow \hat{f}_2(\mathbf{u})$, the convolution of f_1 and f_2 , $f_3(\mathbf{r}) = [f_1 * f_2](\mathbf{r})$ has FT given by:

$$\hat{f}_3(\mathbf{u}) = \hat{f}_1(\mathbf{u}) \cdot \hat{f}_2(\mathbf{u})$$

In other words, *N-D convolution in the spatial domain becomes multiplication in the frequency domain.* Similarly, *multiplication in the spatial domain becomes N-D convolution in the frequency domain.* This will become critical for understanding sampling and aliasing.

- *Cross-Correlation.* For any FT pairs $f_1(\mathbf{r}) \Leftrightarrow \hat{f}_1(\mathbf{u})$, and $f_2(\mathbf{r}) \Leftrightarrow \hat{f}_2(\mathbf{u})$, the cross-correlation of f_1 and f_2 , $f_3(\mathbf{r}) = \int_{-\infty}^{\infty} \overline{f_1(\mathbf{r}')} \cdot f_2(\mathbf{r} + \mathbf{r}') d\mathbf{r}'$ has FT given by:

$$\hat{f}_3(\mathbf{u}) = \overline{\hat{f}_1(\mathbf{u})} \cdot \hat{f}_2(\mathbf{u})$$

Based on this result, the autocorrelation of $f(\mathbf{r})$, given by $\int_{-\infty}^{\infty} \overline{f(\mathbf{y})} \cdot f(\mathbf{r} + \mathbf{r}') d\mathbf{r}'$ has FT given by $\overline{\hat{f}(\mathbf{u})} \cdot \hat{f}(\mathbf{u}) = |\hat{f}(\mathbf{u})|^2$.

- *Derivative.* For any FT pair $f_0(\mathbf{r}) \Leftrightarrow \hat{f}_0(\mathbf{u})$, the FT of the n^{th} derivative of $f_0(\mathbf{r})$ along r_k , $f_{n,k}(\mathbf{r}) = \frac{\partial^n f_0(\mathbf{r})}{\partial r_k^n}$ is:

$$\hat{f}_{n,k}(\mathbf{u}) = (i2\pi u_k)^n \hat{f}_0(\mathbf{u})$$

7.4 Properties of the N-D Fourier Transform that are Specific to Multiple Dimensions

Additionally, the N-D FT has a few properties that arise specifically in the multi-dimensional case:

- *Spatial transformation* For any signal $f_1(\mathbf{r})$ with FT $\hat{f}_1(\mathbf{u})$, the FT of its spatially transformed version $f_2(\mathbf{r}) = f_1(\mathbf{A}\mathbf{r})$ for any invertible $N \times N$ matrix $\mathbf{A}$, is given by

$$\hat{f}_2(\mathbf{u}) = \frac{1}{|\det \mathbf{A}|} \hat{f}_1(\mathbf{A}^{-T} \mathbf{u})$$

where ‘det’ denotes the matrix determinant, and $\mathbf{A}^{-T} = (\mathbf{A}^{-1})^T$ is the transpose of the inverse matrix. Note that this property is a generalization of the case where each dimension k is simply scaled by a scalar factor a_k , in which case $\mathbf{A}$ is a diagonal matrix with diagonal elements $(a_1, a_2, \dots, a_N)$.

- *Rotation.* One consequence of the previous “spatial transformation” property is that rotation of an image results in the same rotation of its FT. In the 2D case, rotation of a 2D image $f_1(x, y)$ by an angle θ ,

$$f_2(x, y) = f_1(x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta)$$

, results in the same rotation of its Fourier Transform, ie:

$$\hat{f}_2(u, v) = \hat{f}_1(u \cos \theta + v \sin \theta, -u \sin \theta + v \cos \theta)$$

.

- *Separability.* If a function $f(\mathbf{r})$ is separable, ie: $f(\mathbf{r}) = f(r_1)f(r_2) \cdots f(r_N)$, then its N-D FT is also separable

$$\hat{f}(\mathbf{u}) = \hat{f}(u_1)\hat{f}(u_2) \cdots \hat{f}(u_N)$$

- *Projections and the Central Section Theorem.* Let us consider the 2D case for simplicity. For a given function (image) $f(x, y)$, we can define a projection along a certain direction parameterized by the angle θ as:

$$g_\theta(x') = \int_{y'} f(x' \cos \theta - y' \sin \theta, x' \sin \theta + y' \cos \theta) dy'$$

(see Figure [7.1](#) for an illustration). The Central Section Theorem states that the (1D) FT of $g_\theta(x')$ is the same as the radial line of the (2D) FT of $f(x, y)$, $\hat{f}(u, v)$ along the same angle θ . This important result, with implications in tomographic image reconstruction and magnetic resonance imaging, is also known as the Central Slice Theorem or Projection-Slice Theorem.

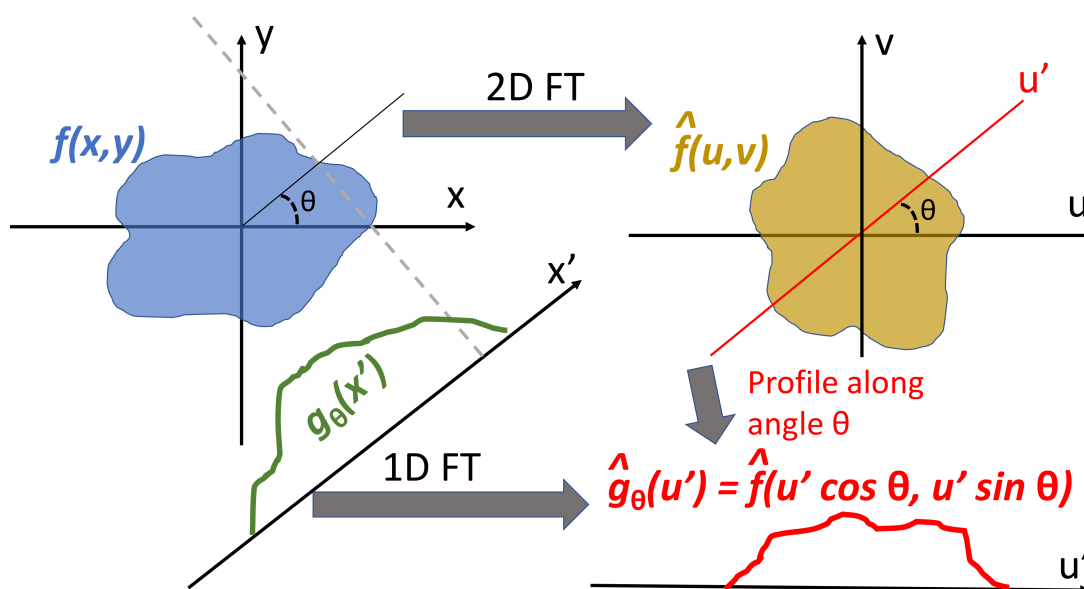


Figure 7.1: Illustration of a 1D projection of a 2D function, and the Central Section Theorem.