Part I

Signal Analysis: Fourier and Wavelet Decompositions

Lecture 1

One- and multi-dimensional signals

1.1 Learning Objectives

- Recap the notation used in this course for one- and multi-dimensional signals, both real-valued and complex-valued.
- Describe two basic types of signals that will be used extensively in this course:
  - Delta functions
  - Complex sinusoids

1.2 Real- and Complex-Valued Signals

1.2.1 Real-Valued Signals

For real-valued signals $f(x)$, the value of $f(x)$ at each location x is a real-valued number, ie: $f(x) \in \mathbb{R}, \forall x$. In general, one might expect to encounter real-valued signals in the real world. However, as we will observe in this course, there is often a clear advantage to representing our signals as complex-valued. For this purpose, we describe complex-valued signals next.

1.2.2 Complex-Valued Signals

For complex-valued signals $f(x)$, the value of $f(x)$ at each location x is generally a complex-valued number, ie: $f(x) \in \mathbb{C}, \forall x$. In other words, at each location x, our signal can be decomposed into real and imaginary parts:

$$f(x) = f_R(x) + i f_I(x) \tag{1.1}$$

where $i = \sqrt{-1}$. Alternatively, this signal can be represented in terms of its magnitude and phase:

$$f(x) = |f(x)| e^{i \angle f(x)} \tag{1.2}$$

where the notation $\angle f(x)$ represents the phase of $f(x)$, in radians. Note that the equivalent notations shown in Equations 1.1 and 1.2 are related as follows:

$$
\begin{aligned}
|f(x)| &= \sqrt{f_R^2(x) + f_I^2(x)} && (1.3)\\
\angle f(x) &= \arctan\left(\frac{f_I(x)}{f_R(x)}\right) \text{ (if } f_R(x) > 0\text{, slightly different otherwise)} && (1.4)\\
f_R(x) &= |f(x)|\cos(\angle f(x)) && (1.5)\\
f_I(x) &= |f(x)|\sin(\angle f(x)) && (1.6)
\end{aligned}
$$

Again, why would we care about complex-valued signals? A straightforward example is Magnetic Resonance Imaging (MRI), where our image is a signal that, at each location, represents a 2D vector of 'transverse magnetization', a location in the x-y plane. This 2D vector $a\hat{x} + b\hat{y}$ can be easily represented as a complex signal where the real part is a and the imaginary part is b. This complex notation makes MRI a lot easier to describe, as will be covered in this and other courses.

1.3 One- and Multi-Dimensional Signals

1.3.1 One-Dimensional Signals

In this course, we will start most sections by describing the 1D version of each concept, which applies to a 1D signal $f(x)$. Note that the definition of x as a spatial location is arbitrary. For instance, all our results apply similarly if we denote our 1D signal as $f(t)$, where t represents time. Of course, reasonable notions of causality may be different for time-domain signal processing, however, this will not affect our mathematical descriptions.

1.3.2 Multi-Dimensional Signals

A simple case of a multi-dimensional signal is a 2D image $f(x, y)$ defined over the x-y plane. Of course, in medical image science we are often interested in 3D images $f(x, y, z)$, or more generally, $f(\boldsymbol{r})$ where $\boldsymbol{r} = [r_1, r_2, \ldots, r_N]$ represents a location in an N-dimensional space.

Question: Can you think of signals that are defined in a space with more than 3 dimensions ($N > 3$)?

1.4 A Couple Important Functions

1.4.1 The Delta (Impulse) Function

Delta Function in 1D

A delta (a generalized function also known as “Dirac delta function”, or “impulse function”) is a very useful construct for signal processing, defined by the properties:

$$\delta(x) = 0 \text{ , for } x \neq 0 \tag{1.7}$$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \tag{1.8}$$

In other words, $\delta(x)$ is zero everywhere except at the origin, where it can be viewed as approaching infinity. The delta function can also be viewed as the limit of of a function in the shape of a ’bump’ as the bump becomes infinitely narrow and tall.[1]

The delta function has a number of important properties. Here we review the properties that are most important for this course:

- *Scaling property.* If we squeeze our delta function by a factor a, we obtain: $\delta(ax) = \frac{\delta(x)}{|a|}$ (ie: a scaled up/down version of the delta function). Among other implications, this leads to the following relationship:

$$\int_{-\infty}^{\infty} \delta(ax)dx = \frac{1}{|a|}$$

- *Sifting property.* The integral of a delta shifted by x_0 times an arbitrary function $f(x)$ is equal to sampling $f(x)$ at $x = x_0$:

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0)$$

  In other words, we can use a train of shifted delta functions to express the sampling of a signal $f(x)$.

- *Comb function.* We can express an infinite string of equally spaced delta functions as

$$Ш(x) = \sum_{m=-\infty}^{\infty} \delta(x - m)$$

  Note that this comb function will become very important when we study sampling.

[1] For details on this formulation, and the variety of limit-based definitions of $\delta(x)$, see for instance http://functions.wolfram.com/GeneralizedFunctions/DiracDelta/09/

Delta Function in N-D

The extension of the delta function to the multi-dimensional case $\delta(x, y)$ or $\delta(\boldsymbol{r})$ is straightforward.

$$\begin{aligned} \delta(\boldsymbol{r}) &= \mathbf{0} \text{ , for } \boldsymbol{r} \neq 0 & (1.9) \\ \int_{\mathbb{R}^N} \delta(\boldsymbol{r}) d\boldsymbol{r} &= 1 & (1.10) \end{aligned}$$

The N-D delta function can be viewed as a product of 1D delta functions along each of the N dimensions:

$$\delta(\boldsymbol{r}) = \delta(r_1) \cdot \delta(r_2) \cdots \delta(r_N) \tag{1.11}$$

The N-D delta function has analogous properties to the 1D version:

- *Scaling property.* If we scale (stretch) our delta function by a factor a_n, $n = 1, 2, \ldots, N$ along each of the N dimensions, we obtain: $\delta(a_1 r_1, a_2 r_2, \ldots, a_N r_N) = \frac{\delta(\boldsymbol{r})}{|a_1 \cdot a_2 \cdots a_N|}$. Among other implications, this leads to the following relationship:

$$\int_{\mathbb{R}^N} \delta(a_1 r_1, a_2 r_2, \ldots, a_N r_N) d\boldsymbol{r} = \frac{1}{|a_1 \cdot a_2 \cdots a_N|}$$

- *Sifting property.* The integral of a delta shifted by $\boldsymbol{r_0}$ times an arbitrary function $f(\boldsymbol{r})$ is equal to sampling $f(\boldsymbol{r})$ at $\boldsymbol{r} = \boldsymbol{r_0}$:

$$\int_{\mathbb{R}^N} f(\boldsymbol{r}) \delta(\boldsymbol{r} - \boldsymbol{r_0}) d\boldsymbol{r} = f(\boldsymbol{r_0})$$

  In other words, we can use a N-D grid of shifted delta functions to express the sampling of a signal $f(\boldsymbol{r})$.

- *Comb function.* We can express an infinite grid of equally spaced delta functions as

$$Ш(\boldsymbol{r}) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \cdots \sum_{m_N=-\infty}^{\infty} \delta(r_1 - m_1, r_2 - m_2, \ldots, r_N - m_N)$$

  Similarly to the 1D case, the N-D comb function will be important when we study multi-dimensional sampling.

1.4.2 The Complex Sinusoid

Complex Sinusoid in 1D

The complex sinusoid signal is defined as:

$$f(x) = A e^{i(2\pi u_0 x + \phi_0)} \tag{1.12}$$

where A is the signal's amplitude, u_0 is the signal's frequency, in cycles per unit length, and ϕ_0 is the initial phase, in radians. Importantly, the complex sinusoid can be described in terms of sines and cosines, as follows:

$$Ae^{i(2\pi u_0 x+\phi_0)} = A\cos(2\pi u_0 x + \phi_0) + iA\sin(2\pi u_0 x + \phi_0) \tag{1.13}$$

Similarly, we can express 'standard' (real) sinusoids in terms of two complex sinusoids with positive and negative frequencies, respectively:

$$A\cos(2\pi u_0 x + \phi_0) = \frac{A}{2}e^{i(2\pi u_0 x+\phi_0)} + \frac{A}{2}e^{-i(2\pi u_0 x+\phi_0)} \tag{1.14}$$

$$A\sin(2\pi u_0 x + \phi_0) = \frac{A}{2i}e^{i(2\pi u_0 x+\phi_0)} - \frac{A}{2i}e^{-i(2\pi u_0 x+\phi_0)} \tag{1.15}$$

$$\tag{1.16}$$

Complex Sinusoid in N-D

We can extend the definition of the complex sinusoid to the N-D case, simply by including an N-D frequency $\boldsymbol{u_0} = [u_{0,1}, u_{0,2}, \ldots, u_{0,N}]$ that describes the propagation of waves in an N-D plane. The complex exponential (assuming zero initial phase, for simplicity), can be described as:

$$f(\mathbf{r}) = Ae^{i2\pi\boldsymbol{u_0}\cdot\boldsymbol{r}} = A\cos(2\pi\boldsymbol{u_0}\cdot\boldsymbol{r}) + iA\sin(2\pi\boldsymbol{u_0}\cdot\boldsymbol{r}) \tag{1.17}$$

where $\boldsymbol{u_0}\cdot\boldsymbol{r}$ denotes scalar (dot) product, ie: $\boldsymbol{u_0}\cdot\boldsymbol{r} = u_{0,1}r_1 + u_{0,2}r_2 + \cdots + u_{0,N}r_N$.

The equivalence between complex sinusoid notation and real sinusoid notation in N-D is analogous to the 1D case described above.

1.5 A Note About Time vs Space

Throughout most of this course, we will assume that our signals are defined in space, eg: $f(x)$ for 1D, $f(r_1, r_2)$ or $f(x, y)$ for 2D, etc, where x, y, r_1, r_2 are units of space along some dimension. This is appropriate for a course focused on image science. However, occasionally during this course the concepts will be better suited for description in terms of signals in time, eg: $f(t)$. In these occasions, we will switch our notation to describe signals in time. However, note that the underlying math and concepts will remain unchanged regardless of the space in which we view our signals.