Lecture 2

LSI systems and convolution in 1D

2.1 Learning Objectives

- Understand the concept of a linear system.
- Understand the concept of a shift-invariant system.
- Recognize that systems that are both linear and shift-invariant can be described simply as a convolution with a certain "impulse response" function.

2.2 Linear Systems

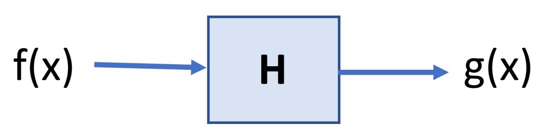

Figure 2.1: Example system H, which takes in an input signal $f(x)$ and produces an output $g(x)$.

A system H takes an input signal x and produces an output signal y, which we can express as $H : f(x) \longrightarrow g(x)$. This very general definition encompasses a vast array of different possible systems. A subset of systems of particular relevance to signal processing are linear systems, where if $f_1(x) \longrightarrow g_1(x)$ and $f_2(x) \longrightarrow g_2(x)$, then:

$$a_1 \cdot f_1 + a_2 \cdot f_2 \longrightarrow a_1 \cdot g_1 + a_2 \cdot g_2$$

for any input signals f_1 and f_2 and scalars a_1 and a_2.

In other words, for a linear system, if we feed it a linear combination of inputs, we get the corresponding linear combination of outputs.

Question: Why do we care about linear systems?

Question: Can you think of examples of linear systems?

Question: Can you think of examples of non-linear systems?

2.3 Shift-Invariant Systems

Another subset of systems we care about are shift-invariant systems, where if $f_1 \longrightarrow g_1$ and $f_2(x) = f_1(x - x_0)$ (ie: f_2 is a shifted version of f_1), then:

$$f_2(x) \longrightarrow g_2(x) = g_1(x - x_0)$$

for any input signals f_1 and shift x_0.

In other words, for shift-invariant systems, if we shift the input in time, the output shifts in time accordingly.

Question: Can you think of examples of shift-invariant systems?

2.4 LSI Systems

Linear shift-invariant systems are systems that satisfy both of the properties described above: linearity and shift-invariance.

Question: Which of the following systems are linear? Which are shift-invariant?

A. $g(x) = f(x)$

B. $g(x) = \max(f(x), 0)$, aka rectified linear unit (ReLU) in machine learning lingo (for real-valued signals)

C. $g(x) = |f(x)|$

D. $g(x) = f(x) + 1$

E. $g(x) = f(0)$

F. $g(x) = \int_{-1}^{+1} f(x + x')dx'$

G. $g(x) = \int_{-\infty}^{+\infty} f(x')e^{-i2\pi x x'}dx'$

One important property of linear shift-invariant systems is that the entire system can be described fully in terms of its response to an impulse input, ie: what is the output corresponding to an 'impulse' input $f(x) = \delta(x)$? This response is often denoted as the "impulse response" $h(x)$, and the central property of LSI systems is that if you know the impulse response $h(x)$ for an LSI system then you can easily calculate the response to *any* input by using the convolution operation. This operation is described next.

2.5 Convolution

The convolution is an operation between two signals f_1 and f_2 defined as:

$$[f_1 * f_2](x) = \int_{-\infty}^{+\infty} f_1(x')f_2(x-x')dx' \tag{2.1}$$

How is this related to LSI systems? Let's say we have an LSI system with impulse response $\delta(x) \longrightarrow h(x)$. If we have an arbitrary input f(x), we can rewrite this input using the sifting property of the delta function:

$$f(x) = \int_{-\infty}^{+\infty} f(x')\delta(x-x')dx' \tag{2.2}$$

Because of shift invariance, the output to $\delta(x-x') \longrightarrow h(x-x')$ for any shift x'.

Because of linearity, the output $g(x)$ to an input $f(x)$ will simply be the sum (integral) of the outputs to each of the shifted values, ie:

$$g(x) = \int_{-\infty}^{+\infty} f(x')h(x-x')dx' \tag{2.3}$$

In other words, the output can be expressed as a convolution of the input with the input response:

$$g(x) = (f * h)(x) \tag{2.4}$$

2.5.1 Properties of Convolution

Below we list a few important properties of convolution:

- Commutative: $(f_1 * f_2)(x) = (f_2 * f_1)(x)$
- Associative: $[(f_1 * f_2) * f_3](x) = [f_1 * (f_2 * f_3)](x)$
- Distributive: $[f_1 * (f_2 + f_3)](x) = (f_1 * f_2)(x) + (f_1 * f_3)(x)$

Note that we will add to these properties when we study Fourier Transforms in a few lectures.

Question: Can a convolution $g(x) = f(x) * h(x)$ be 'undone'? In other words, if we know $g(x)$ (the output) and $h(x)$ (the convolution filter), do you think we can generally recover our input signal $f(x)$? Do not worry if you are not sure at this point. The answer to this question will become a lot easier once we cover Fourier Transforms.

Solution to one of the example questions

Is this system linear? Is this shift invariant?

$$g(x) = \int_{-1}^{+1} f(x + x')dx'$$

Clearly linear, based on the properties of integration (you can check for yourself).

Shift invariant? Let's say the input is $f_1(x)$ with corresponding output $g_1(x)$. Now, let's say we have a shifted input $f_2(x) = f_1(x - x_0)$, ie: a shifted version of our original input. In order for this system to be shift invariant, the output to $f_2(x)$ would need to be $g_2(x) = g_1(x - x_0)$. Is this true?

$$\begin{aligned} g_2(x) &= \int_{-1}^{+1} f_2(x + x')dx' \\ &= \int_{-1}^{+1} f_1(x - x_0 + x')dx' \end{aligned}$$

If we change variables $x^{''} = x - x_0$, ie: $x = x^{''} + x_0$, we can write this as

$$\begin{aligned} g_2(x^{''} + x_0) &= \int_{-1}^{+1} f_1(x^{''} + x')dx' \\ &= g_1(x^{''}) \end{aligned}$$

Or, in other words, if we do another variable change $x^{'''} = x^{''} + x_0$, we can rewrite $g_2(x^{'''}) = g_1(x^{'''} - x_0)$. Note that it doesn't matter if we use the symbol $x^{'''}$ or x in the equation, so $g_2(x) = g_1(x - x_0)$ and our system is shift invariant. In other words, this is an LSI system with impulse response $h(x) = 1$ if $|x| < 1$ and 0 otherwise. Knowing this impulse response $h(x)$, note that we can compute the output for any input $f(x)$, as $(f * h)(x)$.