# Lecture 5

# Properties of the 1D Fourier Transform

## 5.1 Learning Objectives

- Understand a few important properties of the Fourier transform, eg: what happens to the Fourier transform of a signal if we perform certain operations on the signal.

## 5.2 Introduction

In this lecture, we will go over a few important properties of the FT. These properties generally relate the behavior of signals in the spatial domain and the behavior of their FT. Many of these properties are extremely useful for the understanding of medical image science. For instance, they will enable characterization of imaging systems, including resolution and artifacts, as well as image filtering. Further, as we will discuss in this and future lectures, the properties of the Fourier Transform (particularly the convolution property) will lead to efficient implementation of convolutions, with important implications for treatment planning as well as imaging.

## 5.3 Basic Properties of the Fourier Transform

- *Linearity.* For any FT pairs $f_1(x) \Leftrightarrow \hat{f}_1(u)$, and $f_2(x) \Leftrightarrow \hat{f}_2(u)$, and complex scalars $a_1$ and $a_2$, the FT of $f_3(x) = a_1 f_1(x) + a_2 f_2(x)$ is given by:

$$\hat{f}_3(u) = a_1 \hat{f}_1(u) + a_2 \hat{f}_2(u)$$

- *Translation.* For any signal $f_1(x)$ with FT $\hat{f}_1(u)$, the FT of its shifted version $f_2(x) = f_1(x - x_0)$ is given by a modulated version of $\hat{f}_1(u)$:

$$\hat{f}_2(u) = e^{-i2\pi x_0 u} \hat{f}_1(u)$$

- *Modulation.* For any signal $f_1(x)$ with FT $\hat{f}_1(u)$, the FT of its modulated version $f_2(x) = e^{i2\pi x u_0} f_1(x)$ is given by a translated version of $\hat{f}_1(u)$:

$$\hat{f}_2(u) = \hat{f}_1(u - u_0)$$

- *Spatial scaling.* For any signal $f_1(x)$ with FT $\hat{f}_1(u)$, the FT of its spatially scaled version $f_2(x) = f_1(ax)$ for any nonzero complex valued scalar $a$, is given by

$$\hat{f}_2(u) = \frac{1}{|a|}\hat{f}_1(\frac{u}{a})$$

- *Conjugation.* For any signal $f_1(x)$ with FT $\hat{f}_1(u)$, the FT of its conjugated version $f_2(x) = \overline{f_1(x)}$ is given by

$$\hat{f}_2(u) = \overline{\hat{f}_1(-u)}$$

- *Parseval's Theorem.* For any FT pairs $f_1(x) \Leftrightarrow \hat{f}_1(u)$, and $f_2(x) \Leftrightarrow \hat{f}_2(u)$ the following equality holds:

$$\int_{-\infty}^{\infty} f_1(x)\overline{f_2(x)}dx = \int_{-\infty}^{\infty} \hat{f}_1(u)\overline{\hat{f}_2(u)}du$$

- *Convolution Theorem.* For any FT pairs $f_1(x) \Leftrightarrow \hat{f}_1(u)$, and $f_2(x) \Leftrightarrow \hat{f}_2(u)$, the convolution of $f_1$ and $f_2$, $f_3(x) = [f_1 * f_2](x)$ has FT given by:

$$\hat{f}_3(u) = \hat{f}_1(u) \cdot \hat{f}_2(u)$$

  In other words, *convolution in the spatial domain becomes multiplication in the frequency domain.* Similarly, *multiplication in the spatial domain becomes convolution in the frequency domain.* This will become critical for understanding sampling, aliasing, and filtering. Further, this property will have important implications for the development of efficient algorithms in treatment planning, imaging, and many other applications. *Question: Can you think of why this may be?.*

- *Cross-Correlation.* For any FT pairs $f_1(x) \Leftrightarrow \hat{f}_1(u)$, and $f_2(x) \Leftrightarrow \hat{f}_2(u)$, the cross-correlation of $f_1$ and $f_2$, $f_3(x) = \int_{-\infty}^{\infty} \overline{f_1(y)} \cdot f_2(x+y)dy$ has FT given by:

$$\hat{f}_3(u) = \overline{\hat{f}_1(u)} \cdot \hat{f}_2(u)$$

  Based on this result, the autocorrelation of $f(x)$, given by $\int_{-\infty}^{\infty} \overline{f(y)} \cdot f(x+y)dy$ has FT given by $\overline{\hat{f}(u)} \cdot \hat{f}(u) = |\hat{f}(u)|^2$.

- *Derivative.* For any FT pair $f_0(x) \Leftrightarrow \hat{f}_0(u)$, the FT of the $n^{th}$ derivative of $f_0(x)$, $f_n(x) = f_0^{(n)}(x) = \frac{d^n f_0(x)}{dx^n}$ is:

$$\hat{f}_n(u) = (i2\pi u)^n \hat{f}_0(u)$$