Lecture 8

Sampling in One Dimension

8.1 Introduction

So far in this course, we have considered continuous functions, eg: $f(x)$ where $x \in \mathbb{R}$. However, in order to record signals as needed for a wide variety of signal processing applications, we need to sample these signals. Sampling is intimately linked to Fourier analysis, which has been covered in the previous few lectures.

Sampling a signal $f(x)$ can be described as follows:

$$f_S[m] = f(m\ \Delta x), \text{ for } m \in \mathbb{N}$$

where Δx is the sampling interval. Note the use of square brackets to denote a function of a discrete variable $m \in \mathbb{N}$, as opposed to the parentheses used for functions over a continuous variable $x \in \mathbb{R}$.

However, for convenience of analysis, throughout most of the following two lectures we will remain in the continuous domain by expressing the sampled function as a train of shifted and scaled deltas, as follows:

$$f_S(x) = \sum_{m=-\infty}^{\infty} f(m\Delta x)\delta(x - m\Delta x)$$

(please see the following subsections for details).

8.2 Recap of Dirac Delta Functions

For reasons that will become clear below, it is convenient to express the sampling operation as a multiplication with a train of deltas, ie: a comb function. Remember that a delta (a generalized function also known as "impulse function" or "Dirac delta function") is a very useful construct for signal processing, defined by the properties:

$$\delta(x) = 0 \text{ , for } x \neq 0$$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

In other words, $\delta(x)$ is zero everywhere except at the origin, where it can be viewed as approaching infinity. The delta function can also be viewed as the limit of of a function in the shape of a 'bump' as the bump becomes infinitely narrow and tall.[1]

The delta function has a number of important properties. Here we review the properties that are most important for sampling:

- *Scaling property.* If we scale (stretch) our delta function by a factor a, we obtain: $\delta(ax) = \frac{\delta(x)}{|a|}$. Among other implications, this leads to the following relationship:

$$\int_{-\infty}^{\infty} \delta(ax)dx = \frac{1}{|a|}$$

- *Sifting property.* The integral of a delta shifted by x_0 times an arbitrary function $f(x)$ is equal to sampling $f(x)$ at $x = x_0$:

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0)$$

  In other words, we can use a train of shifted delta functions to express the sampling of a signal $f(x)$.

- *Comb function.* We can express an infinite string of equally spaced delta functions as

$$Ш(x) = \sum_{m=-\infty}^{\infty} \delta(x - m)$$

  One interesting and useful property of the comb function is that its Fourier transform is also a comb function:

$$Ш(x) \longrightarrow \hat{Ш}(u) = Ш(u)$$

8.3 Description of Sampling: Spatial Domain

The sampled version of $f(x)$, sampled with an interval Δx can be described as:

$$f_S(x) = f(x) \cdot \frac{1}{\Delta x} Ш(\frac{x}{\Delta x})$$

[1]For details on this formulation, and the variety of limit-based definitions of $\delta(x)$, see for instance http://functions.wolfram.com/GeneralizedFunctions/DiracDelta/09/

where the factor $\frac{1}{\Delta x}$ is needed to normalize each delta function in the comb function. This expression can be rewritten as:

$$\begin{aligned} f_S(x) &= f(x) \sum_{m=-\infty}^{\infty} \delta(x - m\Delta x) \\ &= \sum_{m=-\infty}^{\infty} f(m\Delta x)\delta(x - m\Delta x) \end{aligned}$$

In other words, $f_S(x)$ consists of an infinite string of shifted delta functions, each scaled by the corresponding sample of $f(x)$.

8.4 Description of Sampling: Fourier Domain

A multiplication in the spatial domain becomes a convolution in the Fourier domain. Therefore, if $f(x)$ has Fourier Transform $\hat{f}(u)$, we can express the Fourier Transform of $f_S(x)$, $\hat{f}_S(u)$, as follows:

$$\begin{aligned} \hat{f}_S(u) &= \hat{f}(u) * \text{Ш}(\Delta x\, u) \\ &= \hat{f}(u) * \frac{1}{\Delta x} \sum_{m=-\infty}^{\infty} \delta(u - \frac{m}{\Delta x}) \\ &= \frac{1}{\Delta x} \sum_{m=-\infty}^{\infty} \hat{f}(u - \frac{m}{\Delta x}) \end{aligned}$$

In other words, sampling in the spatial (or temporal) domain leads to a periodic replication in the corresponding Fourier domain. See Figure 8.1 for an illustration of this effect.

8.5 Nyquist Sampling Rate

In principle, a continous signal can be perfectly recovered from its samples, as long as there is no "aliasing". Aliasing is the overlapping of replicates in the Fourier domain that will occur if our sampling rate is not high enough (or equivalently, if the sampling interval is not small enough)[2]. In the absence of aliasing, one could recover $f(x)$ from $f_S(x)$ simply by low-pass filtering $f_S(x)$ using a filter with rect-shaped Fourier transform (with the right bandwidth Δu - see below), in order to keep the central replicate within $\hat{f}_S(u)$ while removing all the other replicates. However, in the presence of aliasing, this perfect recovery is generally not possible as many different signals might have led to our aliased sampled signal $f_S(x)$. In other words, when aliasing occurs we have a fundamental

[2] A popular and intuitive example of aliasing is the wagon wheel effect, which arises in the video recording of wheels in moving cars, as demonstrated in a multitude of amusing videos, including: https://www.youtube.com/watch?v=VNftf5qLpiA, https://www.youtube.com/watch?v=QOwzkND_ooU . Here is the Wikipedia page for this effect: https://en.wikipedia.org/wiki/Wagon-wheel_effect

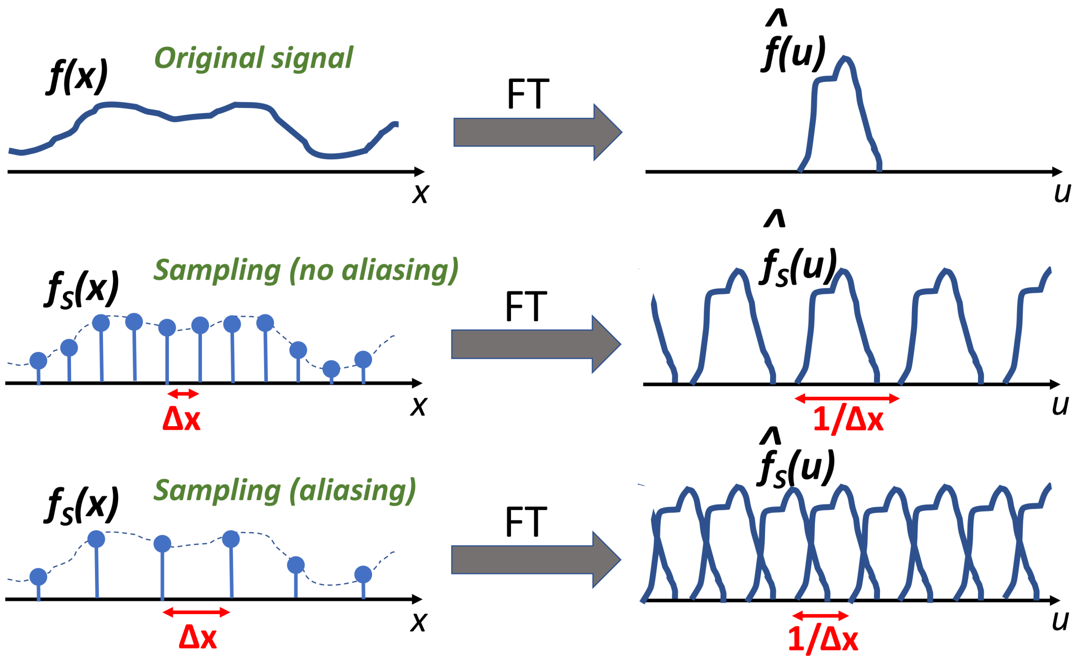

Figure 8.1: Sampling as viewed in the spatial domain as well as in the frequency domain. Sampling a bandlimited signal $f(x)$ (ie: a signal such that $\hat{f}(u)$ has limited support in u) with a small enough sampling interval (ie: with a sampling rate $1/\Delta x$ that is above the Nyquist rate) leads to lack of aliasing in the Fourier domain (ie: the signal can be recovered from the samples). However, sampling with a sampling interval that is too large (ie: with a sampling rate $1/\Delta x$ that is below the Nyquist rate) leads to aliasing (superposition of the Fourier space replicates), and the original signal $f(x)$ cannot be recovered uniquely from the sampled signal $f_S(x)$.

loss of information about our signal, and we can no longer recover it perfectly from our samples. A central question in sampling is, how do we need to sample in order to avoid aliasing?

If a signal $f(x)$ is bandlimited with bandwidth Δu, eg: $\hat{f}(u) = 0$ for $|u| > \Delta u/2$, then its Nyquist rate is Δu. In other words, in order to sample $f(x)$ without aliasing we need to sample with a high enough sampling rate $1/\Delta x > \Delta u$, or equivalently with a small enough sampling interval $\Delta x < 1/\Delta u$. See Figure 8.1 for an illustration.