Lecture 9

Sampling in Multiple Dimensions

9.1 Introduction

Sampling in multiple (two or more) dimensions follows directly from the one-dimensional formulation we have studied. However, the additional dimensions provide extra degrees of freedom with important implications. In this lecture, we will focus on the 2D case (since it is a lot easier to draw, compared to > 2 dimensions).

9.2 Description of Sampling in 2D: Spatial Domain

The sampled version of $f(x, y)$, sampled with an interval Δx along x and Δy along y can be described as:

$$f_S(x, y) = f(x, y) \cdot \frac{1}{\Delta x \Delta x} Ш(\frac{x}{\Delta x}, \frac{y}{\Delta y}) \tag{9.1}$$

where the factor $\frac{1}{\Delta x \Delta y}$ is needed to normalize each delta function in the comb function. This expression can be rewritten as:

$$\begin{aligned} f_S(x, y) &= f(x, y) \cdot \sum_{m,n} \delta(x - m\Delta x, y - n\Delta y) \qquad (9.2) \\ &= \sum_{m,n} f(m\Delta x, n\Delta y)\delta(x - m\Delta x, y - n\Delta y) \qquad (9.3) \end{aligned}$$

In other words, $f_S(x, y)$ consists of an infinite grid of delta functions shifted along x and y, each scaled by the corresponding sample of $f(x, y)$.

9.3 Description of Sampling in 2D: Fourier Domain

Similarly to the 1D case, in multiple dimensions a multiplication in the spatial domain becomes a convolution in the Fourier domain. Therefore, if $f(x, y)$ has Fourier Transform

$\hat{f}(u, v)$, we can express the Fourier Transform of $f_S(x, y)$, $\hat{f}_S(u, v)$, as follows:

$$\begin{aligned}
\hat{f}_S(u, v) &= \hat{f}(u, v) * \text{III}(\Delta x\, u, \Delta y\, v) \\
&= \hat{f}(u, v) * \frac{1}{\Delta x \Delta y} \sum_{m,n} \delta(u - \frac{m}{\Delta x}, v - \frac{n}{\Delta y}) \\
&= \frac{1}{\Delta x \Delta y} \sum_{m,n} \hat{f}(u - \frac{m}{\Delta x}, v - \frac{n}{\Delta y})
\end{aligned}$$

9.4 Nyquist Sampling Rate in 2D

If a signal $f(x, y)$ is bandlimited with bandwidth Δu along x and Δv along y, ie: $\hat{f}(u, v) = 0$ for $|u| > \Delta u/2$ and $\hat{f}(u, v) = 0$ for $|v| > \Delta v/2$, then its Nyquist rate along x is Δu and the Nyquist rate along y us Δv. In other words, in order to sample $f(x, y)$ without aliasing it would suffice to sample with a high enough sampling rate along each dimension: $1/\Delta x > \Delta u$ and $1/\Delta y > \Delta v$. See Figure 8.1 for an illustration.

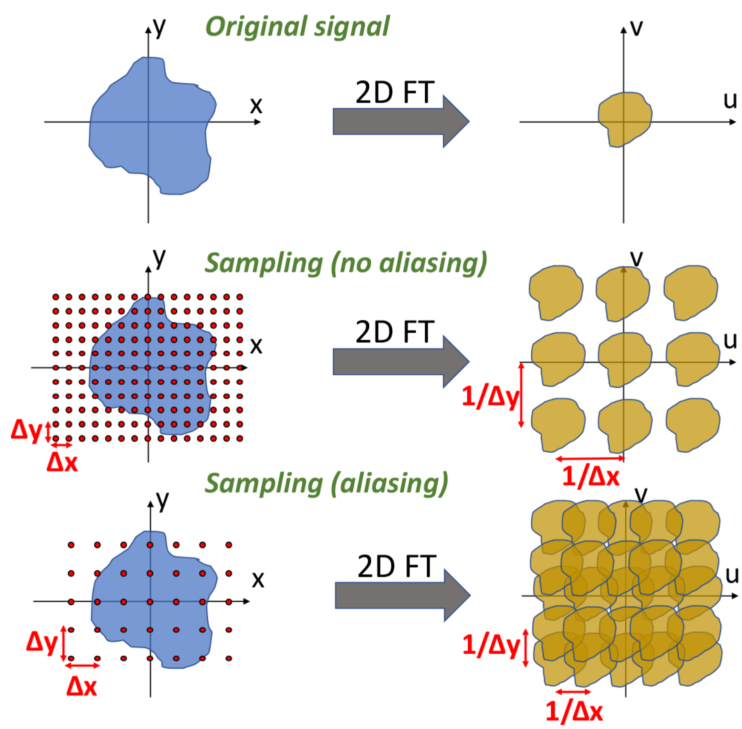

Figure 9.1: Illustration of sampling and aliasing in multiple dimensions.