Lecture 13

Properties of the DFT

13.1 Introduction

In this lecture, we will cover the basic properties of the DFT. Although the presentation will be provided in terms of 1D DFTs, all these properties extend readily to the multi-dimensional case.

13.1.1 Linear vs Circular Convolution

In our discussion of DFTs, it will be important to distinguish the linear convolution (denoted by $*$) from the circular convolution (denoted by $\circledast$) between two sequences.

- *Linear convolution.* For any two length-M sequences $f_1[m]$ and $f_2[m]$, their linear convolution is defined as

$$(f_1 * f_2)[m] = \sum_{l=0}^{M-1} f_1[l] f_2[m-l], \text{ for } m = 0, 1, \ldots, 2M-2$$

  where $f_2[m-l]$ is assumed to be zero for values of $m-l$ outside the range $[0, M-1]$. In other words, the linear convolution assumes a non-periodic (zero-padded) extension of the input sequences.

- *Circular convolution.* For any two length-M sequences $f_1[m]$ and $f_2[m]$, their circular convolution (also known as "periodic convolution") is defined as follows:

$$(f_1 \circledast f_2)[m] = \sum_{l=0}^{M-1} f_1[l] f_2[(m-l)_M], \text{ for } m = 0, 1, \ldots, M-1$$

where "$(\cdot)_M$" in $f_2[(m-l)_M]$ refers to the "modulo" operation (the remainder of the division $\frac{m-l}{M}$), and is sometimes written as $(m-l) \mod M$. Importantly, for two length-M sequences $f_1[m]$ and $f_2[m]$, their linear convolution is a sequence of length $2M-1$, whereas their circular convolution is a sequence of length M. In other words, the linear and circular convolutions are fundamentally different operations. However, they are equivalent under certain conditions, which we will examine in subsequent lectures.

13.2 Properties

- *Linearity.* If $f_1[m]$ and $f_2[m]$ have DFT $\hat{f}_1[k]$ and $\hat{f}_2[k]$, respectively, then $f_3[m] = a_1 f_1[m] + a_2 f_2[m]$ will have DFT $\hat{f}_3[k] = a_1 \hat{f}_1[k] + a_2 \hat{f}_2[k]$.
- *Circular shift.* If length-M sequence $f_1[m]$ has DFT $\hat{f}_1[k]$, then $f_2[m] = f_1[(m-l)_M]$ (ie: f_2 is f_1 circularly shifted by l samples) will have DFT $\hat{f}_2[k] = \hat{f}_1[k] e^{-i2\pi \frac{lk}{M}}$. Note that this is similar to the translation property of the continuous-time FT, except in the case of the DFT it applies to circular shifts rather than linear shifts.
- *Periodicity.* For a length-M sequence $f[m]$ with DFT $\hat{f}[k]$, we might wish to evaluate $\hat{f}[k+M]$ (ie: evaluate $\hat{f}$ beyond the limits we have defined it). However, it is straightforward to show that $\hat{f}[k+M] = \hat{f}[k]$, ie: $\hat{f}$ can be viewed as being length-M or as being periodic with period M. *Question: can you prove this based on the definition of the DFT?*
- *Circular convolution.* The circular convolution between two sequences has DFT given by the multiplication of their corresponding DFTs. In other words, the DFT of $f_3[m] = (f_1 \circledast f_2)[m]$ is $\hat{f}_3[k] = \hat{f}_1[k]\hat{f}_2[k]$.
- *Multiplication.* Analogously to the previous property, the multiplication of two sequences $f_3[m] = f_1[m] f_2[m]$ has DFT given by the *circular convolution* of their corresponding DFTs:

$$\hat{f}_3[k] = \left(\hat{f}_1 \circledast \hat{f}_2\right)[k] = \sum_{n=0}^{M-1} \hat{f}_1[n]\hat{f}_2[(k-n)_M], \text{ for } k = 0, 1, \ldots, M-1$$

- *Complex conjugation.* The DFT of the complex conjugate of a sequence, $f_2[m] = \overline{f_1[m]}$ is the complex conjugate, spatially-reversed version of the original DFT, ie: $\hat{f}_2[k] = \overline{\hat{f}_1[(-k)_M]}$.
- *Space reversal.* The DFT of the reversed version of a sequence, $f_2[m] = f_1[(-m)_M]$, is the reversed version of the original DFT, $\hat{f}_2[k] = \hat{f}_1[(-k)_M]$.
- *Parseval's theorem.* For any two length-M signals $f_1[m]$ and $f_2[m]$, the following relationship holds:

$$\sum_{m=0}^{M-1} f_1[m]\overline{f_2[m]} = \frac{1}{M}\sum_{k=0}^{M-1} \hat{f}_1[k]\overline{\hat{f}_2[k]}$$

  Note that this relationship implies that, given that $f[m]\overline{f[m]} = |f[m]|^2$, the energy of a signal in both domains obeys the following relationship:

$$\sum_{m=0}^{M-1} |f[m]|^2 = \frac{1}{M}\sum_{k=0}^{M-1} |\hat{f}[k]|^2$$