Lecture 18

Wavelets in One Dimension

18.1 Learning Objectives

- Understand the filter bank interpretation of multi-resolution wavelet analysis.
- Recognize basic Matlab commands for wavelet analysis and synthesis.
- Understand a simple denoising application of wavelet analysis.

18.2 DWT and iDWT

The DWT and iDWT are represented as filter banks in figure 18.1.

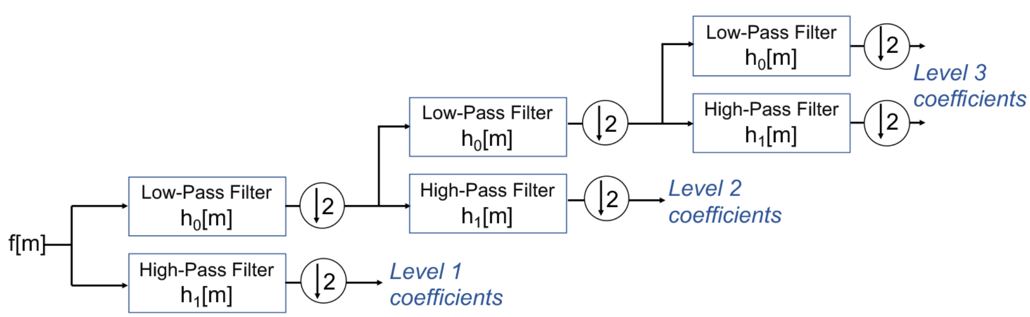

Figure 18.1: Filter bank view of the wavelet transform (with 3 levels).

As shown in Figure 18.2, the DWT can be represented as a cascade of filters, which lead to multi-level coefficients, including high-pass coefficients at multiple levels, as well as a low-pass coefficients at the highest (coarsest) level (see Figure 18.1). Similarly, the iDWT can be represented as a cascade of reconstruction filters, which lead back to reconstruction of the original signal.

In Matlab, a single-level wavelet decomposition can be performed with the command:

```
[cA,cD] = dwt(f,wname);
```

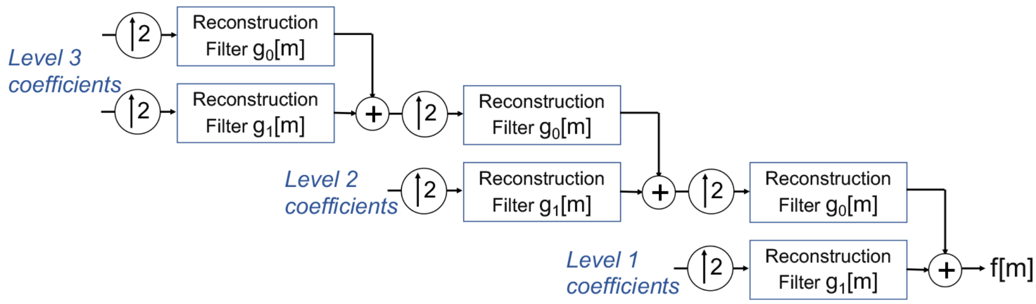

Figure 18.2: Filter bank view of signal synthesis from a wavelet decomposition. For a wide choice of DWTs, the synthesis (or reconstruction) filters are simply the spatially-reversed version of the analysis filters, ie: $g_0[m] = h_0[M-1-m]$, and $g_1[m] = h_1[M-1-m]$. Note that the upsampling operation (up arrow next to the number 2) denotes introducing a sample with value zero after each sample in the input signal.

where `f` is the input signal, `wname` contains the name of the type of wavelets (eg: 'haar', 'db2','db3',...), and `cA` and `cD` contain the approximation coefficients and detail coefficients, respectively. A multi-level wavelet decomposition can be performed using the command `wavedec`. The `dwt` and `wavedec` operations can be inverted to recover the original signal, using the commands `idwt` and `waverec`, respectively.

18.3 DFT vs DWT Decomposition

The DFT and DWT result in very different signal decompositions. As observed in figure 18.3, the DFT of a piecewise constant signal (a profile through the Shepp-Logan digital phantom), the DFT contains heavy signals near at low frequencies, and much lower (but generally non-zero) signals at higher frequencies.

In contrast, the DWT leads to a multi-resolution decomposition where at the coarsest resolution we obtain a 'low-pass' approximation of the signal, and then we obtain progressively finer detail coefficients. In this multi-resolution decomposition, detail coefficients reflect local features of the signal (unlike in the DFT where the basis functions have support over the entire domain). Importantly, for many relevant signals, the DWT results in sparse representations (ie: representations where many of the coefficients are zero). This has implications for a number of applications, including signal processing, denoising, and compression.

18.4 Wavelet-based Denoising

An example application of wavelets vs Fourier analysis is the denoising of signals. Interestingly, denoising is closely related to compression, as it often relies on the sparse representation of a signal as a combination of a few non-zero coefficients, such that all other coefficients can be ignored or set to zero. Denoising can be performed based on

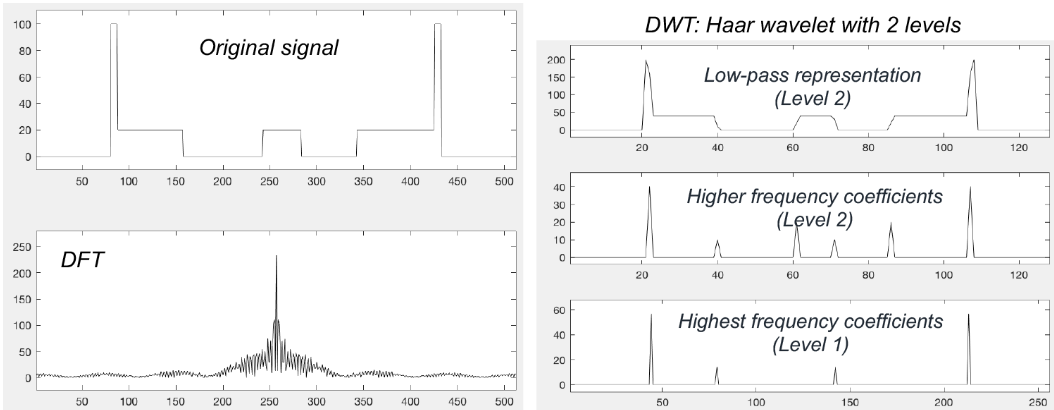

Figure 18.3: DFT vs DWT based decomposition of a piecewise-constant signal. Note that the DFT requires many different frequencies to represent sharp edges, whereas the DWT leads to a sparser representation of sharp edges.

a thresholding (often 'soft'-thresholding) approach in the transform domain, as demonstrated in figure 18.4.

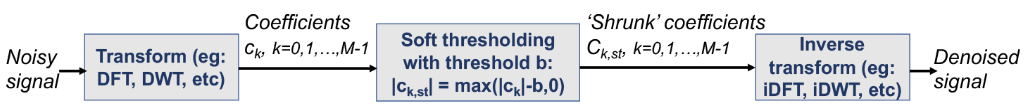

Figure 18.4: Denoising based on soft-thresholding.

In the example shown in Figure 18.5, DWT shows higher performance than DFT-based denoising. Specifically, DFT-based denoising leads to a loss of resolution (as the coefficients that are nulled are typically all the coefficients at high frequencies). In contrast DWT-based denoising is able to reduce noise while maintaining sharp edges.

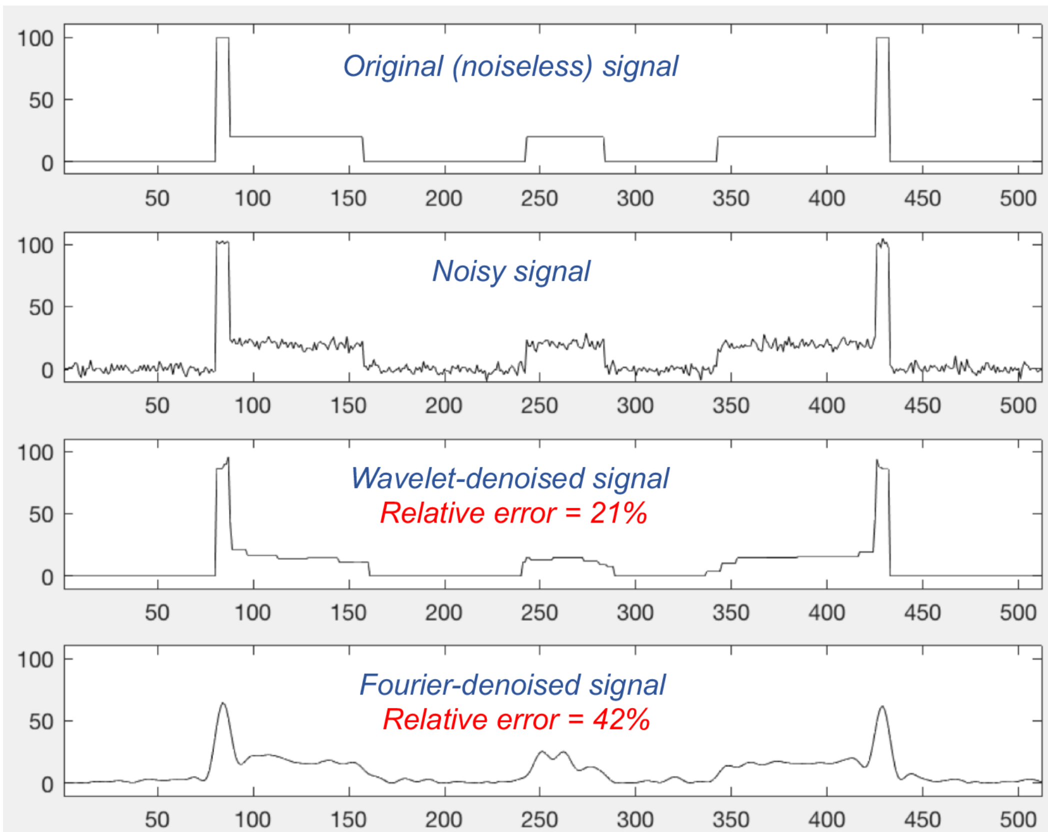

Figure 18.5: DFT vs DWT based denoising of a noisy signal. Note that DFT-based denoising leads to a loss of resolution, as the higher frequency components are removed.